2/13/20

Last Time: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \cdot e^{-t}\, dt$

$\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1)$

$\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha\cdot(\alpha+1)\cdot\dots\cdot(\alpha+k-1)$

Example: $N|\Lambda \sim P(\Lambda)$ & $\Lambda \sim \Gamma(\alpha, \theta)$

Q: $N \sim ?$

A: Determine $f_N(k)$ $\left(= Pr(N=k) \overset{\text{Tables}}{=} p_k\right)$

$f_{N,\Lambda}(k,\lambda) = f_{N|\Lambda=\lambda}(k)\cdot f_\Lambda(\lambda)$

$= \left(\frac{\lambda^k \cdot e^{-\lambda}}{k!}\right)\cdot\left[\frac{1}{\theta^\alpha\Gamma(\alpha)}\cdot\lambda^{\alpha-1}\cdot e^{-\frac{\lambda}{\theta}}\right]$

$\therefore \quad f_N(k) = \frac{1}{k!\,\theta^\alpha\cdot\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha+k-1}\cdot e^{-\lambda(1+\frac{1}{\theta})}\, d\lambda$

Let $t = \lambda(1+\frac{1}{\theta})$

$\begin{array}{c|c} \lambda & t \\ \hline 0 & 0 \\ \infty & \infty \end{array}$

$dt = (1+\frac{1}{\theta})\, d\lambda$

$\lambda = \frac{1}{1+\frac{1}{\theta}}\cdot t = \frac{\theta}{1+\theta}\cdot t$

$= \frac{1}{k!\cdot\theta^\alpha\cdot\Gamma(\alpha)}\int_0^\infty \left(\frac{\theta}{1+\theta}\right)^{\alpha+k-1}\cdot t^{\alpha+k-1}\cdot e^{-t}\cdot\frac{\theta}{1+\theta}\, dt$

$$= \frac{1}{k! \cdot \Theta^{\alpha} \cdot \Gamma(\alpha)} \cdot \left(\frac{\Theta}{1+\Theta}\right)^{\alpha+k} \cdot \underbrace{\int_0^\infty t^{\alpha+k-1} \cdot e^{-t} \cdot dt}_{=\Gamma(\alpha+k)}$$

$$= \frac{\Theta^{k}}{k! \cdot (1+\Theta)^{\alpha+k}} \cdot \underbrace{\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}}_{\alpha \cdot (\alpha+1) \cdot \dots \cdot (\alpha+k-1)}$$

$$\therefore\ f_N(k) = p_k = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1) \cdot \Theta^k}{k!\ (1+\Theta)^{\alpha+k}}$$

$$\Longrightarrow N \sim NB(r=\alpha,\ \beta=\Theta)$$

$$\boxed{\begin{array}{l} \therefore\ N|\Lambda \sim P(\Lambda)\ \ \&\ \ \Lambda \sim \Gamma(\alpha, \Theta) \\ \qquad \Longrightarrow N \sim NB(r=\alpha,\ \beta=\Theta) \end{array}}$$

Continuous Mixture of Continuous Distribution

Example 1)

$$\boxed{\begin{array}{l} X|\Lambda \sim Exp\left(mean = \frac{1}{\Lambda}\right)\ \ \&\ \ \Lambda \sim \Gamma(\alpha, \theta) \\ \qquad \Longrightarrow X \sim 2\text{-Pareto}\left(\alpha' = \alpha,\ \theta' = \frac{1}{\theta}\right) \end{array}}$$

Example 2) $X|\Theta \sim N(\overset{mean}{\Theta}, \overset{variance}{V})\ \ \&\ \ \Theta \sim N(M, A)$

Fact: $X \sim N(\mu, \sigma^2)$ Q: $\mu = ?$ $\sigma^2 = ?$

A: $E[X] = E[E[X|\Theta]] = E[\Theta] = M = \mu$

$\sigma^2 = Var(X) = E[Var(X|\Theta)] + Var(E[X|\Theta]) = E[V] + Var(\Theta) = V + A$

New r.v.'s from old ones

Examples:

1) $Y \sim N(\mu, \sigma^2)$

Define $X = e^Y$ $\quad$ $X \sim \text{Log}N(\mu, \sigma^2)$ (See P.8 of Tables) — log normal

$$F_X(x) = Pr(X \le x) = Pr(e^Y \le x)$$

$$= Pr(Y \le \ln(x)) = Pr\left(SND \le \frac{\ln(x) - \mu}{\sigma}\right)$$

SND — standard normal distribution

$$f_X(x) = f_Y(\ln(x)) \cdot \left|\frac{1}{x}\right|$$

2) Y is given; c - constant

Let $X = Y + c$ (shifted)

$$F_X(x) = Pr(X \le x) = Pr(Y \le x - c) = F_Y(x - c)$$

$$f_X(x) = f_Y(x - c) \cdot |1|$$

3) $Y \sim \text{Blah}(\Theta)$ $\quad$ Θ = scale parameter

Then $X = \frac{1}{Y}$ (Inverse Blah $(\Theta' = \frac{1}{\Theta})$)

$X = Y^{pos}$ (Transformed Blah $(\Theta' = \Theta^{pos})$)

$X = \frac{1}{Y^{pos}}$ (Inverse Transformed Blah $(\Theta' = \frac{1}{\Theta^{pos}})$)